JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **114**, 584–592 (1986)

Uniqueness of Connecting Orbits in the Equation $Y^{(3)} = Y^2 - 1$

CHRISTOPHER K. MCCORD*

Department of Mathematics, University of Wisconsin–Madison, Madison, Wisconsin 53706

Submitted by G.-C. Rota

The differential equation of the title arises (in particular) in the study of shock waves (N. Kopell and L. Howard, *Advan. in Math.* **18** (1975), 306–358). The equation has two rest points; the question is whether there are solutions running between them. Existence of such connecting solutions is proved independantly in C. Conley ("Isolated Invariant Sets and the Morse Index," Cont. Bd. Math. Sci, No. 38, Amer. Math. Soc., Providence, R. I. 1978) and Kopell and Howard (*ibid.*). The uniqueness is proved in this paper. © 1986 Academic Press, Inc.

In [1] and [2], the behavior of bounded solutions of $y^{(2n+1)} = y^2 - 1$, $n \geqslant 0$, is investigated and existence of nonconstant bounded solutions is shown. It is known that any such solution must run from one critical point to the other. We wish to complete this investigation for the case $n = 1$ by showing there is only one nonconstant bounded solution.

The equation $y^{(3)} = y^2 - 1$ can be expressed as the following system of first order equations:

$$
\begin{aligned}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= y_3, \qquad (1)\\
\dot{y}_3 &= y_1^2 - 1.
\end{aligned}
$$

This system has two fixed points, $(1, 0, 0)$ and $(-1, 0, 0)$. The linearized equation at these points has eigenvalues α, $\alpha\omega$, $\alpha\omega^2$ and $-\alpha$, $-\alpha\omega$, $-\alpha\omega^2$, respectively, where $\alpha = \sqrt[3]{2}$, and $\omega = e(2\pi i/3)$. Thus both points are hyperbolic critical points: $(1, 0, 0)$ has a two-dimensional stable manifold and a one-dimensional manifold; $(-1, 0, 0)$ has a one-dimensional stable manifold and a two-dimensional unstable manifold.

This system admits the Liapunov function $L(y) = \frac{1}{3}y_1^3 - y_1 - y_2 y_3$.

* Supported in part by Wisconsin Alumni Research Foundation Fellowship.

0022-247X/86 $3.00

$\dot{L}(y)=-y_3^2$, so L decreases along solutions which do not lie in the $y_3=0$ plane. But a solution can remain in that plane for an interval of time only if $\dot{y}_3=y_1^2-1=0$ throughout the interval, or if $y_1=\pm 1$. And this is only possible if $\dot{y}_1=y_2=0$ throughout the interval. Thus the only solutions on which L is constant are the two fixed points.

Also, this system can be rescaled: if we take

$$x_1=\varepsilon^{1/2}y_1(\varepsilon^{1/6}t),$$
$$x_2=\varepsilon^{2/3}y_2(\varepsilon^{1/6}t),$$
$$x_3=\varepsilon^{5/6}y_3(\varepsilon^{1/6}t),$$

then $x(t)=(x_1(t), x_2(t), x_3(t))$ satisfies

$$\begin{aligned}\dot{x}_1&=x_2,\\ \dot{x}_2&=x_3,\\ \dot{x}_3&=x_1^2-\varepsilon,\end{aligned} \tag{2}$$

with Liapunov function $L(x)=\frac{1}{3}x_1^3-\varepsilon x_1-x_2x_3$. In particular, if $\varepsilon^{1/6}=-1$, then $x(t)=(-y_1(-t), y_2(-t), -y_3(-t))$ satisfies (1). That is, (1) is symmetric under the transformation

$$t\to -t,$$
$$y_1\to -y_1,$$
$$y_2\to y_2,$$
$$y_3\to -y_3,$$

so the rotation about the y_2 axis of a (forward time) solution is itself a solution in backward time.

We wish to find the bounded solutions of (1). Clearly, $(1,0,0)$ and $(-1,0,0)$ are the only fixed points, and there are no periodic solutions, as L decreases along nonconstant solutions. The only other possible bounded solutions are those with ω-limit set $\{(1,0,0)\}$ and α-limit set $\{(-1,0,0)\}$, or vice versa (i.e., connecting orbits). As $L(1,0,0)=-\frac{2}{3}$, $L(-1,0,0)=\frac{2}{3}$, there are no solutions of with α-limit $(1,0,0)$ and ω-limit $(-1,0,0)$. Thus the (qualitative) behavior of bounded solutions of (1) is completely described by the following:

THEOREM. *There exists a unique solution of* (1) *with α-limit* $(-1,0,0)$ *and ω-limit* $(1,0,0)$.

Proof. We will use $y \cdot t$ to denote the point of which y is carried by the flow in time t, $y_i \cdot t$ to denote the ith coordinate of $y \cdot t$, and $y \cdot R$ to denote the orbit through y.

A. Existence

Existence proofs can be found in [1] and [2]. The Morse theory argument found in [1] can be briefly stated here.

Equation (2) for $\varepsilon < 0$ has $\dot{x}_3 = x_1^2 - \varepsilon$ bounded below by $-\varepsilon > 0$, so $x_3(t)$ approaches infinity as t does. Thus the set of bounded orbits is empty when $\varepsilon < 0$. For $\varepsilon = 0$, the only bounded orbit is the fixed point at the origin. Namely, for $\varepsilon = 0$, $x_3(t)$ remains bounded only if the orbit remains in the $\{x_1 = 0\}$ plane for all time. Just as in the argument above for orbits with constant L, this is only possibly if x_2 and x_3 are also zero of all time.

Choose a ball about the origin. For $\varepsilon \leqslant 0$, no boundary point of the ball is on a bounded orbit. It follows that the same is true for small positive ε. So in the sense of [1], the set of bounded orbits in the ball is isolated, and so has a Morse index. Since it continues to the empty set (as ε goes negative) its index is that of the empty set, namely the zero index. On the other hand, when $\varepsilon > 0$, the invariant set in the ball contains the critical points $(\varepsilon^{1/2}, 0, 0)$ and $(-\varepsilon^{1/2}, 0, 0)$. Both of these are isolated, and both have nonzero index. The index of the disjoint union of two isolated invariant sets has zero index if and only if they both do, so $\{(\varepsilon^{1/2}, 0, 0), (-\varepsilon^{1/2}, 0, 0)\}$ has nonzero index. Therefore, the two points cannot make up the full invariant set of bounded orbits when $\varepsilon > 0$. Because of the Liapunov function, all other orbits in the set must connect the two critical points. This proves the existence of connecting orbits.

B. Uniqueness

To prove uniqueness, we proceed by considering intersections of connecting orbits with the plane $Y = \{y_1 = 0\}$. We will show

LEMMA I. *There exists at most one connecting orbit which intersects Y exactly once.*

LEMMA II. *There exist no connecting orbits which intersect Y two or more times.*

In proving these lemmas, the existence of the Liapunov function and the symmetry under rotation about the y_2 axis will be crucial.

Proof of Lemma I. Let $Y_1 = \{y \in Y \mid y \cdot R$ is a connecting orbit, $y \cdot R \cap Y = y\}$. Namely, Y_1 is the intersection of Y with the connecting orbits which pass through Y once. If $y, y' \in Y_1$, then $y_1 \cdot t, y'_1 \cdot t > 0$ for $t > 0$, and their difference $x \cdot t = (y - y') \cdot t$ satisfies

$$\begin{aligned}
\dot{x}_1 &= \dot{y}_1 - \dot{y}'_1 = y_2 - y'_2 = x_2, \\
\dot{x}_2 &= \dot{y}_2 - \dot{y}'_2 = y_3 - y'_3 = x_3, \\
\dot{x}_3 &= \dot{y}_3 - \dot{y}_3 = y_1^2 - y_1'^2 \\
&= (y_1 + y'_1)(y_1 - y'_1) = \Phi(t)\, x_1,
\end{aligned} \tag{3}$$

with $\Phi(t) > 0$ for $t > 0$.

This equation has fixed point (0, 0, 0), and positively invariant cone $X = \{x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0\} - \{(0, 0, 0)\}$. That is, if an x orbit enters this cone, all of its coordinates and all of its derivatives are positive for all forward time, so all of its coordinate functions diverge to infinity.

Now consider $y = (0, y_2, y_3) \in Y$, with $y_3 \geqslant 0$. Because of the rotational symmetry, $y' = (0, y_2, -y_3) \in Y$ also [i.e., $y \cdot R$ has α-limit $(-1, 0, 0)$, so $y' \cdot R$ has ω-limit $(1, 0, 0)$; $y \cdot R$ has ω-limit $(1, 0, 0)$, so y' has α-limit $(-1, 0, 0)$]. Then, as both $y \cdot R$ and $y' \cdot R$ have ω-limit $(1, 0, 0)$, $(y - y') \cdot t \to (0, 0, 0)$ as $t \to \infty$. But $y - y'$ envolves according to (3), so if $y_3 > 0$, $y - y' = (0, 0, 2y_3) \in X$, and $(y - y') \cdot t \to \infty$. Thus $y_3 = 0$, and all elements of Y_1 lie on the y_2 axis.

If $y = (0, y_2, 0)$, $y' = (0, y'_2, 0) \in Y_1$ with $y_2 > y'_2$, then $(y - y') \cdot t$ goes to $(0, 0, 0)$ as $t \to \infty$. But then $y - y' = (0, y_2 - y'_2, 0) \in X$, so $(y - y') \cdot t \to \infty$ as $t \to \infty$. Thus $y_2 = y'_2$, and Y_1 contains at most one element.

Proof of Lemma II. Clearly, an orbit $y \cdot R$ intersects Y twice or more if and only if, at some time t_0, either $y_1 \cdot t$ goes from positive to negative, or $y \cdot R$ is tangent to Y. Equivalently, $y \cdot R$ intersects Y twice or more if and only if, at some time t_0, $y_1 \cdot t_0 = 0$, and $y_2 \cdot t_0 = \dot{y}_1 \cdot t_0 \leqslant 0$. Thus it suffices to show that no point of $Y \cap \{y_2 \leqslant 0\}$ lies in a connecting orbit. As we have shown above that $(0, y_2, y_3)$ lies in a connecting orbit if and only if $(0, y_2, -y_3)$ does, it suffices to show that no point of $U = \{y_1 = 0, y_2 \leqslant 0, y_3 \leqslant 0\}$ lies in a connecting orbit.

To show that this is the case, consider the behavior of solutions of (1) in the following regions:

$$\begin{aligned}
U_1 &= \{-1 \leqslant y_1 \leqslant 0, y_2 \leqslant 0, y_3 \leqslant 0\} - \{(-1, 0, 0)\}, \\
U_2 &= \{y_1 \leqslant -1, y_2 \leqslant 0, y_3 \leqslant 0\} - \{(-1, 0, 0)\}, \\
U_3 &= \{y_1 \leqslant -1, y_2 \leqslant 0, y_3 \geqslant 0\} - \{(-1, 0, 0)\}.
\end{aligned}$$

Then $\dot{y}_1, \dot{y}_2, \dot{y}_3 \leqslant 0$, with at least one of these nonzero, for all points $(y_1,$

$y_2, y_3) \in U_1$, so orbits in U_1 exit U_1 through $U_1 \cap U_2$. In particular, all points of $U = U_1 \cap Y$ are carried by the flow into the interior of U_1, and then into $U_1 \cap U_2 \cap \{y_3 < 0\}$. Points in U_2 have $\dot{y}_1, \dot{y}_2 \leqslant 0$, $\dot{y}_3 \geqslant 0$, so all orbits in U_2 exit U_2 through $U_2 \cap U_3$, and points on $U_1 \cap U_2 \cap \{y_3 < 0\}$ do so through $U_2 \cap U_3 \cap \{y_1 < -1, y_2 < 0\}$. Finally, points in U_3 have $\dot{y}_1 \leqslant 0$, $\dot{y}_2, \dot{y}_3 \geqslant 0$, so orbits in U_3 exit U_3 through $\{y_1 \leqslant -1, y_3 \geqslant 0\}$, with points on $U_2 \cap U_3 \cap \{y_1 < -1, y_2 < 0\}$ exiting U_3 through $\{y_1 < 0, y_3 > 0\}$ (see Fig. 1).

Thus, if $y \in U$, then there exist $0 < t_1 < t_2 < t_3$ such that

$$
\begin{array}{lll}
y_1 \cdot 0 = 0, & y_2 \cdot 0 \leqslant 0, & y_3 \cdot 0 \leqslant 0, \\
y \cdot t \in \operatorname{int} U_1 & \text{for all } 0 \leqslant t \leqslant t_1, & \\
y_1 \cdot t_1 = -1, & y_2 \cdot t_1 < 0, & y_3 \cdot t_1 < 0, \\
y \cdot t \in \operatorname{int} U_2 & \text{for all } t_1 < t < t_2, & \\
y_1 \cdot t_2 < -1, & y_2 \cdot t_2 < 0, & y_3 \cdot t_2 = 0, \\
y \cdot t \in \operatorname{int} U_3 & \text{for all } t_2 < t < t_3, & \\
y_1 \cdot t_3 < -1, & y_2 \cdot t_3 = 0, & y_3 \cdot t_3 > 0.
\end{array}
\tag{4}
$$

We can refine some of these estimates. Since L decreases along solutions, $L(y) > L(y \cdot t_1) > L(y \cdot t_2) > L(y \cdot t_3)$. But for $y \in U$, $L(y) = -y_2 y_3 \leqslant 0$ so $L(y \cdot t_1) = \frac{2}{3} - (y_2 \cdot t_1)(y_3 \cdot t_1) < 0$, $L(y \cdot t_2) = \frac{1}{3}(y_1 \cdot t_2)^3 - y_1 \cdot t_2 < 0$, and $L(y \cdot t_3) = \frac{1}{3}(y_1 \cdot t_3)^3 - y_1 \cdot t_3 < \frac{1}{3}(y_1 \cdot t_2)^3 - y_1 \cdot t_2$. The function $\frac{1}{3}x^3 - x$ is an

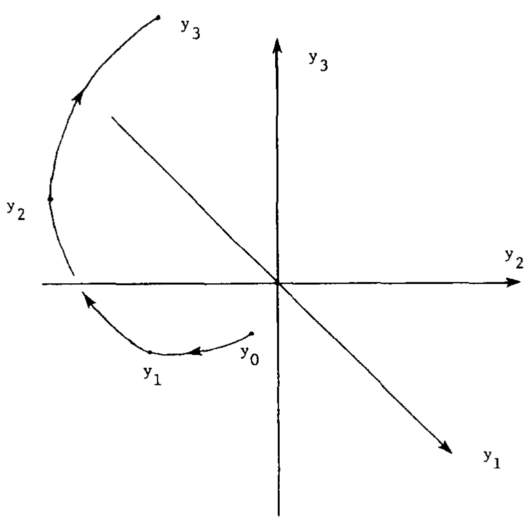

FIG. 1. Orbits through U.

increasing function for $x < -1$, with $\frac{1}{3}\ (-\sqrt{3})^3 + \sqrt{3} = 0$. Thus $y_1 \cdot t_3 < y_1 \cdot t_2 < -\sqrt{3}$, and $(y_2 \cdot t_1)(y_3 \cdot t_1) > \frac{2}{3}$.

In each of U_1, U_2, U_3, all derivatives have constant sign throughout the region, so we can estimate how y_1, y_2, y_3, and L change as orbits run through the regions. For example, $y_1 \cdot t_1 - y_1 \cdot 0 = \int_{[0,t_1]} y_2 \cdot t\, dt$, so

$$\min_{t \in [0,\, t_1]} \{y_2 \cdot t\} \leqslant \frac{y_1 \cdot t_1 - y_1 \cdot 0}{t_1 - 0} \leqslant \max_{t \in [0,\, t_1]} \{y_2 \cdot t\}.$$

But y_2 is decreasing in U_1, and we have chosen $y_1 \cdot 0 = 0$, $y_1 \cdot t_1 = -1$, so this inequality becomes $y_2 \cdot t_1 \leqslant -1/t_1 \leqslant y_2 \cdot 0$. Further, in U_1, $\ddot{y}_2 = \dot{y}_3 = y_1^2 - 1 \leqslant 0$, so y_2 is concave down as a function of t, and we can sharpen our estimate to $\frac{1}{2}(y_2 \cdot 0 + y_2 \cdot t_1) \leqslant -1/t_1 \leqslant y_2 \cdot 0$.

We can repeat this argument for y_1, y_2, y_3, and L in U_1, U_2, and U_3. If we do so, using the convexity argument wherever possible, putting in known values of $y_i \cdot t_j$, and multiplying inequalities by -1 wherever appropriate to obtain nonnegative inqualities, we have

$$-y_{20} \leqslant \frac{1}{t_1} \leqslant -\frac{1}{2}(y_{20} + y_{21}), \tag{5a}$$

$$-\frac{1}{2}(y_{30} + y_{31}) \leqslant \frac{y_{20} - y_{21}}{t_1} \leqslant -y_{31}, \tag{5b}$$

$$\frac{1}{2} \leqslant \frac{y_{30} - y_{31}}{t_1} \leqslant 1, \tag{5c}$$

$$y_{30}^2 \leqslant \frac{L_0 - L_1}{t_1} \leqslant y_{31}^2, \tag{5d}$$

$$-\frac{1}{2}(y_{21} + y_{22}) \leqslant \frac{-1 - y_{12}}{t_2 - t_1} \leqslant -y_{22}, \tag{5e}$$

$$-\frac{1}{2} y_{31} \leqslant \frac{y_{21} - y_{22}}{t_2 - t_1} \leqslant -y_{31}, \tag{5f}$$

$$0 \leqslant \frac{-y_{31}}{t_2 - t_1} \leqslant \frac{1}{2}(y_{12}^2 - 1), \tag{5g}$$

$$0 \leqslant \frac{L_1 - L_2}{t_2 - t_1} \leqslant y_{31}^2, \tag{5h}$$

$$-\frac{1}{2} y_{22} \leqslant \frac{y_{12} - y_{13}}{t_3 - t_2} \leqslant -y_{22}, \tag{5i}$$

$$0 \leqslant \frac{-y_{22}}{t_3 - t_2} \leqslant \frac{1}{2} y_{33}, \tag{5j}$$

$$y_{12}^2 - 1 \leqslant \frac{y_{33}}{t_3 - t_2} \leqslant y_{13}^2 - 1, \tag{5k}$$

$$0 \leqslant \frac{L_2 - L_3}{t_3 - t_2} \leqslant \frac{1}{2} y_{33}^2, \tag{5l}$$

where $y_{ij} = y_i \cdot t_j$, and $L_j = L(y \cdot t_j)$. (Inequalities c, g, h and l will not be used in the lemma proof, but are included for completeness.)

With this system of inequalities, we can complete the lemma proof with the following:

PROPOSITION. *Let $y \cdot R$ be a solution of* (1), *with $y \in U$, and $y_i \cdot t_j$ as in* (4). *Then $y_1 \cdot t_3 \leqslant -2$.*

With this proposition, we see that for any $y \in U$, $L(y \cdot t_3) = \frac{1}{3}(y_1 \cdot t_3)^3 - y_1 \cdot t_3 \leqslant \frac{1}{3}(-2)^3 + 2 = -\frac{2}{3}$. In particular, $L(y \cdot t_3) \leqslant L(1, 0, 0)$, so $y \cdot t \nrightarrow (1, 0, 0)$ as $t \to \infty$, and $y \cdot R$ is not a connecting orbit.

We prove the proposition by contradiction. Assume that there exists a $y \in U$ with $y \cdot t$ as in (4), and $y_{13} > -2$. Then y_{ij} satisfy (5), with $-\sqrt{3} > y_{13} > y_{12} > -2$. Also, as $-\frac{2}{3} = \frac{1}{3}(-2)^3 + 2 < L_3 < L_2 < L_1 < L_0 \leqslant 0$, we have $-\frac{2}{3} < \frac{2}{3} - y_{21}y_{31} = L_1 < 0$, or $\frac{2}{3} < y_{21}y_{31} < \frac{4}{3}$. Similarly, $0 < y_{20}y_{30} < \frac{2}{3}$. From this we find

(i) $-y_{22} < \frac{4}{5}$

Proof. From (5i)–(5k), we have

$$-\frac{1}{2} y_{22} < \frac{y_{12} - y_{13}}{t_3 - t_2} < \frac{2 - \sqrt{3}}{t_3 - t_2},$$

and

$$-y_{22} < \frac{1}{2} y_{33}(t_3 - t_2) \leqslant \frac{1}{2} (y_{13}^2 - 1)(t_3 - t_2)^2 \leqslant \frac{3}{2} (t_3 - t_2)^2.$$

Thus $-y_{22} \leqslant \min\{2(2 - \sqrt{3})(t_3 - t_2)^{-1}, \frac{3}{2}(t_3 - t_2)^2\}$.

The functions $2(2 - \sqrt{3})\, x^{-1}$ and $\frac{3}{2}x^2$ are equal at $x = [\frac{4}{3}(2 - \sqrt{3})]^{1/3}$. For $x > 0$, x^2 is strictly increasing, x^{-1} is strictly decreasing, so the minimum of the two functions at each x is less than or equal to the value of the functions at their intersection. Thus, $-y_{22} \leqslant \frac{3}{2}[\frac{4}{3}(2 - \sqrt{3})]^{2/3} < \frac{4}{5}$.

(ii) $-y_{20} < \frac{4}{5}$, $-y_{30} < \frac{3}{4}$, $t_1 > \frac{5}{4}$.

Proof. As $y_2 \cdot t$ is decreasing for $0 < t < t_2$, $y_{20} > y_{21} > y_{22} > -\frac{4}{5}$. Thus

$-y_{20}<\frac{4}{5}$, and $-\frac{1}{2}(y_{20}+y_{21})<\frac{4}{5}$, so by (5a), $\frac{5}{4}<2/-(y_{20}+y_{21})\leqslant t_1$. Then by (5d),

$$y_{30}^2\leqslant\frac{L_0-L_1}{t_1}<\frac{4}{5}(L_0-L_1)<\frac{4}{5}\frac{2}{3}=\frac{8}{15},\quad \text{as } 0>L_0>L_1-\frac{2}{3}.$$

So $-y_{30}<\sqrt{8}/\sqrt{15}<\frac{3}{4}$.

(iii) $\frac{5}{6}<-y_{31}<\frac{32}{25}$.

Proof. $-y_{21}<\frac{4}{5}$, and $(-y_{21})(-y_{31})>\frac{2}{3}$, so $-y_{31}>2/3(-y_{21})>\frac{2}{3}\frac{5}{4}=\frac{5}{6}$. Also, from (ii) and (5b),

$$-\frac{1}{2}y_{31}\leqslant-\frac{1}{2}(y_{30}+y_{31})\leqslant\frac{y_{20}-y_{21}}{t_1}<\frac{4}{5}(y_{20}-y_{21}).$$

But $0<-y_{20}<-y_{21}<\frac{4}{5}$, so $y_{20}-y_{21}<\frac{4}{5}$. Thus $-y_{31}<2(\frac{4}{5})(\frac{4}{5})=\frac{32}{25}$.

(iv) $\frac{7}{8}<t_2-t_1<\frac{48}{25}$.

Proof. $\sqrt{3}<-y_{12}$, so $\sqrt{3}-1<-1-y_{12}$. Then (i) and (5e) give

$$\frac{3-1}{t_2-t_1}<\frac{-1-y_{12}}{t_2-t_1}<-y_{22}<\frac{4}{5}.$$

Then $\frac{7}{8}<\frac{5}{4}(\sqrt{3}-1)<t_2-t_1$. Also, as $0<-y_{21}<-y_{22}<\frac{4}{5}$, $y_{21}-y_{22}<\frac{4}{5}$, so (5f) gives

$$\frac{4}{5(t_2-t_1)}>\frac{y_{21}-y_{22}}{t_2-t_1}>-\frac{1}{2}y_{31}.$$

Then, from (iii) we have

$$\frac{4}{5(t_2-t_1)}>\frac{5}{12}\qquad\text{or}\qquad t_2-t_1<\frac{48}{25}.$$

(v) $\frac{25}{48}<-y_{21}<\frac{21}{48}$.

Proof. $(-y_{21})(-y_{31})>\frac{2}{3}$, so (iii) implies that

$$-y_{21}>\frac{2}{3(-y_{31})}>\frac{2}{3}\frac{25}{32}=\frac{25}{48}.$$

But (5f) implies $-\frac{1}{2}y_{31}(t_2-t_1)\leqslant y_{21}-y_{22}$, and (iii) and (iv) give $\frac{35}{96}=\frac{1}{2}\frac{5}{6}\frac{7}{8}<y_{21}-y_{22}$. Thus $-y_{21}<-y_{22}-\frac{35}{96}<\frac{4}{5}-\frac{35}{96}<\frac{21}{48}$.

This gives the contradiction $25<21$, so the assumption that there exists a $y\in U$ with $y_1\cdot t_3>-2$ must be false.

Acknowledgment

I am indebted to Charles Conley for proposing this problem to me, and for his useful suggestions on its proof.

References

1. C. Conley, "Isolated invariant sets and the Morse Index," Conf. Bd. Math. Sci., No. 38, Amer. Math. Soc., Providence, R. I., 1978.
2. N. Kopell and L. Howard, Bifurcations and trajectories joining critical points, *Adv. in Math.* **18** (1975), 306–358.